

Quantization from Hamilton-Jacobi theory with a random constraint

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We propose a method of quantization based on Hamilton-Jacobi theory in the presence of a random constraint due to the fluctuations of a set of hidden random variables. Given a Lagrangian, it reproduces the results of canonical quantization yet with a unique ordering of operators if the Lagrange multiplier that arises in the dynamical system with constraint can only take binary values $\pm\hbar/2$ with equal probability.

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I. MOTIVATION

In the previous work [1], we have developed a new method of quantization of system of spin-less particles based on a specific modification of the classical dynamics of ensemble of trajectories parameterized by an unbiased (hidden) random variable. The fluctuations of the random variable is then characterized by the Planck constant. Given a wide class of classical Hamiltonians, the quantization is done by assuming rules of replacement of (deterministic) c-number by (stochastically parameterized) c-number to be applied to the classical Hamilton-Jacobi and continuity equations generated by the classical Hamiltonian. Assuming a specific type of distribution of the random variable, the modified Hamilton-Jacobi and continuity equations are then written into the Schrödinger equation with a unique quantum Hamiltonian, thus is free from the operator ordering ambiguity of the canonical quantization [2]. Moreover, one can always identify an effective velocity field which turns out to be equal to the actual velocity field of the particles in pilot-wave theory [3].

In the present paper, we shall show that the rules of replacement heuristically postulated in Ref. [1] can be derived from the Hamilton-Jacobi theory with a random constraint. The fluctuations of the constraint is assumed due to the presence of some background fields whose detail interaction with the particles are not known. The existence of background fields is also assumed in the Nelson stochastic mechanics to explain the origin of quantum fluctuations as time symmetric Brownian motion [4]. We shall then show that the hidden random variable postulated in Ref. [1] can be identified as the Lagrange multiplier that arises in the corresponding dynamical system with random constraint.

II. HAMILTON-JACOBI THEORY WITH RANDOM CONSTRAINT

Let us first review the Hamilton-Jacobi theory of classical dynamics. For simplicity, below we shall discuss the case of system with finite degrees of freedom. Let us denote the corresponding Lagrangian of the system as

$\underline{L}(q, \dot{q})$ where $q = \{q_i\}$, $i = 1, 2, \dots$, runs for all degrees of freedom, is the configuration coordinate, t is time and $\dot{q} \doteq dq/dt = \{\dot{q}_i\}$ is the velocity. The canonical conjugate momentum is defined as

$$\underline{p}_i(q, \dot{q}; t) \doteq \frac{\partial \underline{L}}{\partial \dot{q}_i}. \quad (1)$$

If the Lagrangian is not singular, $\det(\partial^2 \underline{L} / \partial \dot{q}_i \partial \dot{q}_j) \neq 0$, which is assumed to avoid unnecessary complication, then the above equation can be solved in term of \dot{q} as

$$\dot{q}_i = \dot{q}_i(q, \underline{p}; t). \quad (2)$$

Now let us define action as $\underline{I} \doteq \int \underline{L}(q, \dot{q}) dt$. The actual trajectory connecting two spacetime points is then obtained by extremizing \underline{I} with respect to variations of (q, \dot{q}) for a pair of fixed ends $\delta \underline{I} = \delta \int_{q_1(t_1)}^{q_2(t_2)} \underline{L}(q, \dot{q}) dt = 0$. This is the Hamilton's principle of stationary action which leads directly to the Euler-Lagrange and Hamilton equations.

Another way to solve the above variational problem is through the Hamilton-Jacobi method [5]. First we construct a one-parameter family of hypersurfaces $\underline{S}(q; t) = \underline{\tau}$, where $\underline{\tau}$ is some parameter labeling the hypersurfaces, so that any point in the configuration space belongs only to one of the surfaces. Moreover a trajectory crosses each hypersurface only once and is nowhere tangent to it. $\underline{\tau}$ is thus a function of time, $\underline{\tau} = \underline{\tau}(t)$, so that one has

$$\underline{\Delta} \doteq \frac{d\underline{\tau}}{dt} = \frac{d\underline{S}}{dt} = \partial_t \underline{S} + \dot{q} \cdot \partial_q \underline{S}. \quad (3)$$

By construction we have $\underline{\Delta} \neq 0$. The Hamilton-Jacobi method to get the actual trajectory then proceeds as follows. Let us consider two hypersurfaces $\underline{S} = \underline{\tau}$ and $\underline{S} = \underline{\tau} + d\underline{\tau}$, separated by infinitesimal $d\underline{\tau}$. Then given any point on the hypersurface $\underline{S} = \underline{\tau}$, the actual trajectory that passes through this point and reaches the surface of $\underline{S} = \underline{\tau} + d\underline{\tau}$ is the one with a velocity \dot{q} that minimizes $d\underline{I}/d\underline{\tau}$ [5]. Hence, the actual velocity must solve the following necessary Hamilton-Jacobi condition:

$$\frac{\partial}{\partial \dot{q}_i} \left(\frac{d\underline{I}}{d\underline{\tau}} \right) = 0, \quad (4)$$

with fixed $d\tau$. Since $dL/d\tau = \underline{L}/\underline{\Delta}$ and $\underline{\Delta} \neq 0$, Eq. (4) then reduces into

$$\frac{\partial \underline{L}}{\partial \dot{q}_i} = \frac{\underline{L}}{\underline{\Delta}} \partial_{q_i} \underline{S}, \quad (5)$$

where we have used $\partial \underline{\Delta}/\partial \dot{q}_i = \partial_{q_i} \underline{S}$ from Eq. (3).

Next, assuming that $\underline{L}/\underline{\Delta} = \theta(\underline{\tau})$, where θ is a function only of $\underline{\tau}$, then one can reparametrize \underline{S} so that $\underline{L} = \underline{\Delta}$ [5]. In this case, one gets

$$\underline{p}_i = \frac{\partial \underline{L}}{\partial \dot{q}_i} = \partial_{q_i} \underline{S}. \quad (6)$$

Inserting this into Eq. (2) one thus has

$$\dot{q}_i = \dot{q}_i(q, \partial_q \underline{S}; t). \quad (7)$$

Finally, substituting Eq. (7) back into Eq. (3), recalling $\underline{L} = \underline{\Delta}$, one obtains the Hamilton-Jacobi equation

$$\partial_t \underline{S} + \dot{q}(q, \partial_q \underline{S}; t) \cdot \partial_q \underline{S} - \underline{L}(q, \dot{q}(q, \partial_q \underline{S}; t)) = 0. \quad (8)$$

One can further show that the Hamilton-Jacobi equation of (8) is equivalent to the Euler-Lagrange and the Hamilton equation [5]. In contrast to the latter two equations, the Hamilton-Jacobi equation describes a congruence of trajectories whose velocity field is given by Eq. (7). A single trajectory is then obtained if one also fixes the initial configuration of the system. Another important feature of Hamilton-Jacobi theory is that it imposes a local condition to the actual trajectory, that of Eq. (4). By contrast, Hamilton's principle gives a global condition to the actual trajectory.

Now let us assume that the system under consideration depends on a set of hidden random variables $\xi \doteq (\xi_1, \xi_2, \dots)$ whose dynamical origin is not known. This for example might be due to the presence of background fields whose detail interaction with the particles is not known resulting in a stochastic motion of the latter. Single event is thus inherently random. Hence, one can only make prediction concerning an ensemble of copies of the system. Let us denote the joint-probability density of the fluctuations of q and ξ as $\Omega(q, \xi; t)$. The marginal probability densities of the configuration of the system q and ξ are then given as

$$\rho(q; t) \doteq \int d\xi \Omega(q, \xi; t) \quad \& \quad P(\xi) \doteq \int dq \Omega(q, \xi; t). \quad (9)$$

Here we have assumed that the probability density of ξ is independent of time. Let us further assume that there is a random velocity field $v(q, \xi; t)$, of the same dimension as the system, so that $\Omega(q, \xi; t)$ has to satisfy the following differential equation:

$$G(q, \dot{q}; \Omega, v) = \frac{d(\ln \Omega)}{dt} + \partial_q \cdot v = \frac{\partial_t \Omega}{\Omega} + \frac{\partial_q \Omega}{\Omega} \cdot \dot{q} + \partial_q \cdot v = 0. \quad (10)$$

The above constraint might be interpreted that the velocity divergence gives the only source of (local in configuration space) change of entropy.

Next let us construct a new family of hypersurfaces $S(q, \xi; t) = \tau$ so that for a fixed value of ξ , any point in configuration space belongs to only one of the hypersurfaces. Yet, a single point can belong to more than one hypersurfaces with different values of ξ . Namely $S(q, \xi; t)$ is now fluctuating due to the fluctuations of ξ . Further, let us assume that for a fixed value of ξ , a trajectory satisfying Eq. (10) crosses a hypersurface only once and nowhere tangent to it. τ is thus a function of t and ξ , $\tau = \tau(t, \xi)$. Now, let us again consider two hypersurfaces $S = \tau$ and $S = \tau + d\tau$ separated by infinitesimal $d\tau$. Let us assume that within this interval, ξ is fixed. Then, as in the case with no constraint discussed before, let us postulate that given any point on the hypersurface $S = \tau$, the actual trajectory that passes through this point and reaches the hypersurface $S = \tau + d\tau$ is the one with a velocity \dot{q} that minimizes $dL/d\tau$ and satisfying the constraint of Eq. (10). We thus have to solve Eq. (4) with the constraint of Eq. (10). Using the Lagrange method, this problem implies the following necessary condition:

$$\frac{\partial}{\partial \dot{q}_i} \left(\frac{dL}{d\tau} \right) + \lambda(\xi) \frac{\partial G}{\partial \dot{q}_i} = 0, \quad (11)$$

where $\lambda = \lambda(\xi)$ is the Lagrange multiplier.

Notice that since the constraint is fluctuating randomly due to the fluctuations of ξ , then the Lagrange multiplier also depends on the value of the hidden variables thus is inherently random. Let us denote the probability density of λ as $P(\lambda)$. As is clear from Eq. (11), ξ appears explicitly in the equation only through $\lambda(\xi)$. Accordingly, we shall regard λ as the effective hidden random variable and use it in place of ξ . Hence, for example, we shall write $\Omega(q, \lambda; t)$ instead of $\Omega(q, \xi; t)$ and so on. Further, by construction ξ thus λ in general depends on the configuration space and time. Later we shall assume that the derivatives of λ with respect to space and time are negligible as compared to that of S . The value of λ can be obtained by inserting Eq. (11) back into Eq. (10). λ thus depends on Ω and v . Below we shall go the other way around. Namely, we shall assume λ with a specific statistical properties and look for a class of Ω and v that satisfy Eqs. (10) and (11).

One can then proceed as before to arrive at the following pair of equations [6]:

$$\partial_{q_i} S = \frac{\partial L}{\partial \dot{q}_i}, \quad \partial_t S + \frac{\partial L}{\partial \dot{q}} \cdot \dot{q} - L = 0. \quad (12)$$

where $L = L(q, \dot{q}, \lambda; \Omega, v)$ is an extended Lagrangian defined as

$$\begin{aligned} L(q, \dot{q}, \lambda; \Omega, v) &\doteq \underline{L}(q, \dot{q}) - \lambda G(q, \dot{q}; \Omega, v) \\ &= \underline{L}(q, \dot{q}; t) - \lambda \left(\frac{\partial_t \Omega}{\Omega} + \frac{\partial_q \Omega}{\Omega} \cdot \dot{q} + \partial_q \cdot v \right). \end{aligned} \quad (13)$$

Equation (12) has to be solved together with the constraint of Eq. (10). From the left equation of (12), using Eq. (13), one gets

$$p_i(q, \dot{q}, \lambda; t, \Omega) \doteq \frac{\partial L}{\partial \dot{q}_i} = \underline{p}_i(q, \dot{q}; t) - \lambda \frac{\partial_{q_i} \Omega}{\Omega} = \partial_{q_i} S. \quad (14)$$

Again, assuming that the unconstrained Lagrangian \underline{L} is not singular, the above equation can be solved in term of \dot{q} to give

$$\dot{q}_i = \dot{q}_i(q, \underline{p}; t) = \dot{q}(q, \partial_q S + \lambda \frac{\partial_q \Omega}{\Omega}; t). \quad (15)$$

This velocity field implies the following continuity equation conserving the probability:

$$\partial_t \Omega + \partial_q \cdot \left(\dot{q}(q, \partial_q S + \lambda \frac{\partial_q \Omega}{\Omega}; t) \Omega \right) = 0. \quad (16)$$

Further, inserting Eq. (15) into the right equation of (12), one obtains the following (modified) Hamilton-Jacobi equation:

$$\begin{aligned} & \partial_t S + \dot{q}(q, \partial_q S + \lambda \frac{\partial_q \Omega}{\Omega}; t) \cdot \partial_q S \\ & - L(q, \dot{q}(q, \partial_q S + \lambda \frac{\partial_q \Omega}{\Omega}; t), \lambda; \Omega, v) = 0, \end{aligned} \quad (17)$$

where we have used the left equation of (12).

We have thus Eqs. (16) and (17) that have to be solved in term of Ω and S subjected to the condition of Eq. (10). They are coupled to each other since \dot{q} now depends on S and Ω . We shall show in the next section by taking a concrete example that Eqs. (16) and (17) can be combined together to give a single partial differential equation for S and Ω [7]. The combined equation has to be solved with the condition of Eq. (10). To do this, we thus have to express v that appears in both equations as function of S and Ω . On the other hand, to be meaningful, the constraint has to be consistent with the dynamics. Namely, if initially Ω satisfies the constraint of Eq. (10) then it must be so for any time as Ω is evolved by the dynamics through Eq. (16). This naturally implies that the random velocity field of the constraint v has to be related to velocity field of the dynamics \dot{q} .

Now let us assume that $v(q, \lambda; t)$ is given as

$$v_i(q, \lambda; t) \doteq \frac{\dot{q}_i(q, \lambda; t) + \dot{q}_i(q, -\lambda; t)}{2} = v_i(q, -\lambda; t). \quad (18)$$

$v(q, \lambda; t)$ is thus uniquely determined by the choice of unconstrained Lagrangian \underline{L} . Moreover, let us further assume a class of solutions satisfying the following symmetry relations:

$$S(q, \lambda; t) = S(q, -\lambda; t) + S_0(\lambda) \quad \& \quad \Omega(q, \lambda; t) = \Omega(q, -\lambda; t), \quad (19)$$

where $S_0(\lambda)$ is independent of q and t . The former can be done by choosing an appropriate parameterization of the hypersurfaces, namely $\tau(\lambda, t) = \tau(-\lambda, t) + S_0(\lambda)$. Moreover, the latter implies that λ is an unbiased random variable

$$P(\lambda) = \int dq \Omega(q, \lambda; t) = P(-\lambda). \quad (20)$$

Let us show that the above choice of random velocity field v generates a constraint that is consistent with the

dynamics. To see this, first, notice that taking the case when λ is positive in the constraint of Eq. (10) add to it the case when λ is negative and divided by two, imposing Eqs. (18) and (19), one gets

$$\partial_t \Omega + \partial_q \cdot (v \Omega) = 0, \quad (21)$$

which is just a continuity equation. Hence, in this case, the constraint is just a probability conservation equation generated by random velocity field v . On the other hand, from Eq. (16), taking the case when λ is positive add to it the case when λ is negative and divided by two one gets, by virtue of Eqs. (18) and (19),

$$\partial_t \Omega + \partial_q \cdot (v \Omega) = 0, \quad (22)$$

which is the same as Eq. (21). Hence the constraint is consistent with the dynamics, as expected.

III. “EFFECTIVE” PILOT-WAVE MODEL

For simplicity, let us apply the general formalism developed in the previous section to ensemble of system of single particle of mass m subjected to external potentials. The unconstrained (classical) Lagrangian then takes the form

$$\underline{L}(q, \dot{q}) = \frac{1}{2} m \dot{q}^2 + A(q) \cdot \dot{q} - V(q), \quad (23)$$

so that one has

$$\underline{p}_i = m \dot{q}_i + A_i. \quad (24)$$

The classical Hamiltonian reads

$$\underline{H} = \frac{1}{2m} (\underline{p} - A)^2 + V. \quad (25)$$

Inserting Eq. (24) into Eq. (14) one gets

$$\dot{q}_i = \frac{\partial_{q_i} S}{m} - \frac{A_i}{m} + \frac{\lambda}{m} \frac{\partial_{q_i} \Omega}{\Omega}. \quad (26)$$

Hence, Eq. (16) becomes

$$\partial_t \Omega + \frac{1}{m} \partial_q \cdot ((\partial_q S - A) \Omega) + \frac{\lambda}{m} \partial_q^2 \Omega = 0. \quad (27)$$

Next, substituting Eq. (26) into Eq. (18) and imposing the assumption of Eq. (19), v is given by

$$v_i(q, \lambda; t) = \frac{1}{m} (\partial_{q_i} S - A_i). \quad (28)$$

Inserting Eqs. (13), (23), (26) and (28) into Eq. (17) one obtains

$$\begin{aligned} & \partial_t S + \frac{(\partial_q S - A)^2}{2m} + V - \frac{2\lambda^2}{m} \frac{\partial_q^2 R}{R} \\ & + \frac{\lambda}{\Omega} \left(\partial_t \Omega + \frac{1}{m} \partial_q \cdot ((\partial_q S - A) \Omega) + \frac{\lambda}{m} \partial_q^2 \Omega \right) = 0, \end{aligned} \quad (29)$$

where we have defined $R \doteq \sqrt{\Omega}$ and used the following identity:

$$\frac{1}{4} \frac{\partial_{q_i} \Omega \partial_{q_j} \Omega}{\Omega^2} = \frac{1}{2} \frac{\partial_{q_i} \partial_{q_j} \Omega}{\Omega} - \frac{\partial_{q_i} \partial_{q_j} R}{R}. \quad (30)$$

Substituting Eq. (27), Eq. (29) then reduces into

$$\partial_t S + \frac{(\partial_q S - A)^2}{2m} + V - \frac{2\lambda^2}{m} \frac{\partial_q^2 R}{R} = 0. \quad (31)$$

Moreover, in this case, inserting Eq. (28) into Eq. (21), the constraint then reads

$$\partial_t \Omega + \frac{1}{m} \partial_q \cdot ((\partial_q S - A)\Omega) = 0. \quad (32)$$

We have thus pair of coupled Eqs. (31) and (32) parameterized by the fluctuating Lagrange multiplier $\lambda(\xi)$. Now, since λ is non-vanishing, one can define the following complex-valued function:

$$\Psi(q, \lambda; t) \doteq R \exp\left(\frac{i}{2|\lambda|} S\right). \quad (33)$$

It differs from the Madelung transformation in that S is divided by $2|\lambda|$ instead of \hbar so that one has $\rho(q; t) = \int d\lambda \Omega = \int d\lambda |\Psi|^2$. The pair of Eqs. (31) and (32) can then be recast into the following modified Schrödinger equation parameterized by the fluctuating Lagrange multiplier:

$$i2|\lambda| \partial_t \Psi = \frac{1}{2m} (-i2|\lambda| \partial_q - A)^2 \Psi + V \Psi, \quad (34)$$

where we have imposed the assumption that the space and time derivatives of λ are negligible as compared to that of S .

Let us proceed to assume that Ω is factorisable as

$$\Omega(q, \lambda; t) = \rho(q, |\lambda|; t) P(\lambda), \quad (35)$$

with $\int dq \rho(q, |\lambda|; t) = 1$ for arbitrary value of λ . This guarantees that Ω is correctly normalized $\int d\lambda dq \Omega = 1$. Moreover, let us assume that the Lagrange multiplier can only take binary values $\lambda(\xi) = \pm \hbar/2$ with equal probability

$$P(\lambda) = \frac{1}{2} \delta(\lambda - \hbar/2) + \frac{1}{2} \delta(\lambda + \hbar/2). \quad (36)$$

In this case, Eq. (34) becomes

$$i\hbar \partial_t \Psi_Q = \frac{1}{2m} (-i\hbar \partial_q - A)^2 \Psi_Q + V \Psi_Q, \quad (37)$$

where the wave function $\Psi_Q(q; t)$ is given by

$$\Psi_Q(q; t) \doteq \sqrt{\rho(q, \hbar/2; t)} \exp\left(\frac{i}{\hbar} S(q, \pm \hbar/2; t)\right). \quad (38)$$

Hence $\rho(q; t) = |\Psi_Q(q; t)|^2$ holds by construction, and the phase is given by $S_Q(q; t) \doteq S(q, \pm \hbar/2; t)$.

The quantum mechanical Schrödinger equation is thus reproduced as a specific case of the present statistical model when the Lagrange multiplier λ is an unbiased binary random variable which can only take values $\lambda(\xi) = \pm \hbar/2$. To this end, it is interesting to mention Ref. [8] which showed that the master equation of a particle moving with a fixed velocity, imposed to a random complete reverse of direction following a Poisson distribution, can be written into Dirac equation (in the same way that the Schrödinger equation is connected to the dynamics of Brownian motion) through analytic continuation. Note also that while λ is a binary random variable, it is a function of the true hidden random variables ξ which in turn may take continuous values. For example one may have $\lambda = \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2} = \pm \hbar$ so that ξ lies on the surface of a ball of radius \hbar . If we divide the surface of the ball into two, attribute each deviation $\pm \hbar$, respectively, and assume that ξ moves sufficiently chaotic, then one will have $\lambda = \pm \hbar$ with equal probability.

Moreover, in this case, the velocity field v that generates the constraint then becomes

$$\begin{aligned} v_i(q, \pm \hbar/2; t) &= \frac{1}{m} (\partial_{q_i} S(q, \pm \hbar/2; t) - A_i) \\ &= \frac{1}{m} (\partial_{q_i} S_Q(q; t) - A_i) \doteq \tilde{v}_i(q; t). \end{aligned} \quad (39)$$

Keeping this in mind, Eqs. (21) or (32) then reads

$$\partial_t \rho + \frac{1}{m} \partial_q \cdot ((\partial_q S_Q - A)\rho) = 0, \quad (40)$$

where we have used Eq. (35). It turns out that \tilde{v} defined above is numerically equal to the actual velocity of the particle in pilot-wave theory [3]. This is also equal to the “naively observable velocity field” reported in Ref. [9], obtained using the notion of weak measurement [10] within the standard interpretation of quantum mechanics.

We have thus an “effectively” similar picture with pilot-wave theory in the sense that the particle always possesses definite position and momentum and further it moves “as if” it is guided by the wave function so that the “effective velocity” \tilde{v} is given by Eq. (39). Hence, we can conclude that the statistical model developed in the present paper will reproduce the statistical wave-like interference pattern in slits experiment and tunneling over potential barrier [11]. It is then tempting to further investigate in the future whether the model can lead to a description of quantum measurement which solves the infamous measurement problem. Note that to discuss the problem of measurement one needs to consider the time-irreversible process of registration which involves realistic description of apparatus and bath with large degrees of freedom [12].

However, in contrast to pilot-wave theory, as is clearly shown in Eq. (39), \tilde{v} is not the actual velocity of the particle, but the average of two actual velocities corresponding to Lagrange multiplier equal to $\hbar/2$ and $-\hbar/2$.

Hence, while pilot-wave theory is strictly deterministic, the present model is inherently stochastic. Further, in this statistical model, rather than being postulated, the Schrödinger equation and a unique guidance relation emerge naturally by imposing a random constraint to the Hamilton-Jacobi condition. It is also evident that in contrast to pilot-wave theory, the wave function in the model is not physically real. It is just an artificial mathematical tool to describe the dynamics and statistics of the ensemble of trajectories, and by construction the Born's statistics $\rho(q; t) = |\Psi_Q(q; t)|^2$ is valid for all time. The so-called quantum potential given by the last term of Eq. (31) with $\lambda = \pm\hbar/2$, which is argued by pilot-wave theory to be responsible for all peculiar quantum phenomena [3], is generated by the local change in configuration space of the entropy part of the constraint. Note however that since the Schrödinger equation is time-reversal invariant then the total change of entropy should be vanishing. Finally while pilot-wave theory can deal with a single trajectory (since the wave function is assumed to be physically real satisfying the deterministic Schrödinger equation), the present model strictly concerns the dynamics of an ensemble of trajectories as in Nelson stochastic mechanics.

IV. QUANTIZATION WITH UNIQUE ORDERING AND DIRECT PHYSICAL INTERPRETATION

We have shown in the previous section by taking an example of particle in external potentials how to develop from a given classical Lagrangian a Schrödinger equation with unique quantum Hamiltonian. It can thus be regarded as to provide a method of quantization of classical system given its classical Hamiltonian. In contrast to the canonical quantization which is formal-mathematical, the method presented in the previous section is based on Hamilton-Jacobi theory in the presence of random constraint so that the quantum-classical correspondence is physically kept transparent.

To develop further formal comparison with canonical quantization, now let us extract some rules which can be applied directly given a classical Hamiltonian. First, let us define a scalar function \underline{S} so that $d\underline{S} = \partial_t \underline{S} dt + \partial_q \underline{S} dq = \underline{L} dt$. Then, recalling that $dS = \partial_t S dt + \partial_q S dq = L dt$, from Eq. (13), we have the following pair of relations:

$$\partial_q \underline{S} = \partial_q S + \lambda \frac{\partial_q \Omega}{\Omega}, \quad \partial_t \underline{S} = \partial_t S + \lambda \frac{\partial_t \Omega}{\Omega} + \lambda \partial_q \cdot v. \quad (41)$$

On the other hand, the Hamilton-Jacobi equation of (17) can be rewritten as

$$\partial_t S + \lambda \frac{\partial_t \Omega}{\Omega} + \lambda \partial_q \cdot v + \dot{q} \cdot \left(\partial_q S + \lambda \frac{\partial_q \Omega}{\Omega} \right) - \underline{L} = 0. \quad (42)$$

Applying Eq. (41), the above equation thus becomes

$$\partial_t \underline{S} + \underline{H}(q, \underline{p})|_{\underline{p}=\partial_q \underline{S}} = 0, \quad (43)$$

where $\underline{H}(q, \underline{p}) = \dot{q} \cdot \underline{p} - \underline{L}$ has the same form as classical Hamiltonian. Next, applying the left equation of (41), the velocity field of Eq. (15) also becomes

$$\dot{q} = \dot{q}(q, \underline{p}; t)|_{\underline{p}=\partial_q \underline{S}} = \frac{\partial \underline{H}}{\partial \underline{p}} \Big|_{\underline{p}=\partial_q \underline{S}}, \quad (44)$$

so that the continuity equation of Eq. (16) now reads

$$\partial_t \Omega + \partial_q \cdot \left(\Omega \frac{\partial \underline{H}}{\partial \underline{p}} \Big|_{\underline{p}=\partial_q \underline{S}} \right) = 0. \quad (45)$$

Notice then that Eqs. (43) and (45) take the same form as the Hamilton-Jacobi equation and the continuity equation of classical mechanics given a classical Hamiltonian $\underline{H}(q, \underline{p})$. Hence, given the classical Hamiltonian, to get the Hamilton-Jacobi equation of (17) and continuity equation of (16) based on which we derive the Schrödinger equation, we can first develop the corresponding classical equations of (43) and (45), and apply the rules of Eq. (41). The remaining task is to express v defined in Eq. (18) in term of the classical Hamiltonian \underline{H} . To do this, one can see that if the classical Hamiltonian is at-most-quadratic in classical momentum, then \dot{q} is a linear function of $\underline{p} = \partial_q \underline{S} = \partial_q S + \lambda(\partial_q \Omega/\Omega)$. In this case, evaluating Eq. (18) and taking into account Eq. (19) one then obtains

$$\begin{aligned} v &= \frac{\dot{q}(q, \partial_q S + \lambda \frac{\partial_q \Omega}{\Omega}; t) + \dot{q}(q, \partial_q S - \lambda \frac{\partial_q \Omega}{\Omega}; t)}{2} \\ &= \dot{q}(q, \partial_q S; t) = \frac{\partial \underline{H}}{\partial \underline{p}} \Big|_{\underline{p}=\partial_q S}. \end{aligned} \quad (46)$$

Inserting Eq. (46) into the constraint of Eq. (21), one thus has

$$\partial_t \Omega + \partial_q \cdot \left(\Omega \frac{\partial \underline{H}}{\partial \underline{p}} \Big|_{\underline{p}=\partial_q S} \right) = 0. \quad (47)$$

It is then imperative to check whether the above equation is consistent with Eq. (45). To see this, notice that v in Eq. (46) is obtained by averaging $\dot{q}(\pm\lambda)$. Hence, Eq. (47) has to be obtained by averaging Eq. (45) for the case of $\pm\lambda$ as well. One can evidently see by taking into account Eq. (19) that for classical Hamiltonian at-most-quadratic in momentum, this is indeed the case. Namely, the constraint of Eq. (47) is indeed consistent with Eq. (45).

Given the classical Hamiltonian, we have thus two rules of Eqs. (41) and (46) to be applied to the classical mechanical equations of (43) and (45) and proceed in the way described in the previous section to arrive at the Schrödinger equation with a unique Hermitian quantum Hamiltonian. The above derived rules are just the rules of quantization proposed heuristically in Ref. [1], that is Eq. (5) of Ref. [1], where formal “replacement” there is re-interpreted in the present paper as physical “substitution” [13]. Hence, we have given a justification of the rules postulated in Ref. [1] in term Hamilton-Jacobi

condition with a random constraint. In other words, the hidden random variable λ postulated in Ref. [1] is given physical interpretation as a random Lagrange multiplier that arises in Hamilton-Jacobi theory with a randomly fluctuating constraint.

V. CONCLUSION AND DISCUSSION

We have proposed a statistical model of quantization given a classical Lagrangian by assuming the existence of hidden random variables and accordingly imposing the Hamilton-Jacobi condition for the actual trajectory to a random constraint due to the fluctuations of hidden variables. Quantum fluctuations is shown to be emergent corresponding to a specific constraint depending uniquely on the choice of the classical Lagrangian and assuming that the Lagrange multiplier that arises in the dynamical system with constraint, which is fluctuating due to the fluctuations of the constraint, can only take binary values $\pm\hbar/2$ with equal probability. Given a classical system, the model leads to a unique quantum system with a straightforward physical interpretation.

Let us mention several interesting problems that can

be raised within the present statistical model. First, it is imperative to ask if the model can suggest new testable predictions beyond quantum mechanics. Such prediction, obtained by allowing fluctuations of $|\lambda|$ around $\hbar/2$ with very small yet finite width, is reported in Ref. [14]. Next, while we have offered an explanation on the physical origin of Planck constant in term of Lagrange multiplier, there is still a missing explanation on what determines its numerical value. Such a question of course beyond the standard quantum mechanics and is permissible only within a model in which quantum fluctuations is emergent [15]. Finally, notice that in the model, the random velocity constraint v is uniquely determined by the choice of classical (unconstrained) Lagrangian. In this sense, the random constraint is already inherent in or self-generated by the system being constrained. Hence, to each quantum system there is a *hidden context* specific to the system.

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